Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 10: Functions. Inverses and compositions. Section 2.3

1 Inverses and compositions

Definition 1. Let f be a **one-to-one correspondence** from the set A to the set B. The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Definition 2. Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition** of the functions f and g, denoted by $f \circ g$ is defined by

$$(f \circ g)(a) = f(g(a)) \qquad \text{for all} \qquad a \in A.$$

Example 3. Consider $f(x) = 2x - 5$ and $g(x) = 3x + 2$ both as functions $\mathbb{R} \to \mathbb{R}$. Find $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = 2(3x + 2) - 5 = 6x - 1 \qquad (g \circ f)(x) = 3(2x - 5) + 2 = 6x - 13.$$

As we can see, the compositions $f \circ g$ and $g \circ f$ do not need to be equal.

Example 4. Let $f\colon A \to B$ be a bijection and $f^{-1}\colon B \to A$ the inverse map. Then

$$f \circ f^{-1} = \iota_B \qquad \text{and} \qquad f^{-1} \circ f = \iota_A.$$

Example 5. As seen before, the function $h\colon \{1, 2, 3\} \to \{a, b, c\}$ defined by

$$h(1) = a, \quad h(2) = b \quad \text{and} \quad h(3) = c,$$

is both injective and surjective. The inverse function is $h^{-1}\colon \{a, b, c\} \to \{1, 2, 3\}$ assigning

$$h(a) = 1, \quad h(b) = 2 \quad \text{and} \quad h(c) = 3.$$

Example 6. Consider the function $f_3\colon [0, \infty) \to [0, \infty)$ given by the formula $f_3(x) = x^2$. We saw already that this is a bijective function and therefore there is an inverse

$$f_3^{-1}\colon [0, \infty) \to [0, \infty)$$

given by $f_3(x) = \sqrt{x}$.